The Investigation of Triangle Inequality

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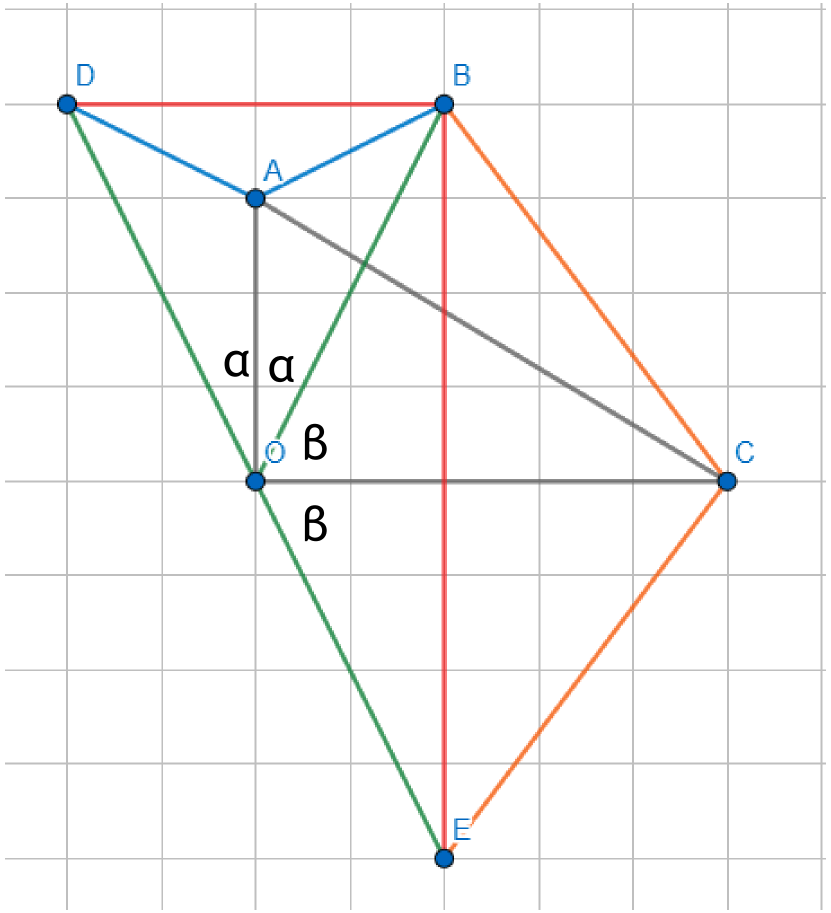
Criminal Geometry, or a Matter of Principle

- Interested in doing something geometry related and came across a Sherlock Holmes articles which also greatly interests me.
- Article on discussion of geometric problems by Sherlock Holmes & Dr. John Watson (Criminal Geometry or a Matter of Principle)
- 4 problems in the article (completed)
- 5 problems from the newspaper, which is discussed at the end of the article (completed)
- After completing the 9 problems, I discovered that quite a few of the problems were solved using triangle inequalities
  - 1 example: Problem #2 from the newspaper

Problem #2 from Newspaper

- Point B lies inside a right angle with vertex O, points A and C lie on its two sides. Prove that the perimeter of triangle ABC is not less than twice the length of segment OB.



WTS: $\boldsymbol{P\Delta ABC > 2(OB)}$

- If you reflect point B over the axes, you get points D & E.
- Also, $< DOA \cong < BOA \text{ and } < BOC \cong < EOC$ (due to reflections)
- $\Delta OBC \cong \Delta OEC \text{ by } SAS \rightarrow OB = OE$
- $\Delta DOA \cong \Delta BOA \text{ by } SAS \rightarrow OD = OB$
- Since point O represents the origin and $< AOB = \alpha \text{ and } < COB = \beta \text{ then } \alpha + \beta = 90º$
- $< DOE = 2\alpha + 2\beta \ \& \ \alpha + \beta = 90º \text{ then } < DOE = 180º$
  - $DE < DA + AC + CE$
  - $2(OB) < DA + AC + CE$
  - $\boldsymbol{2(OB) < BA + AC + BC}$

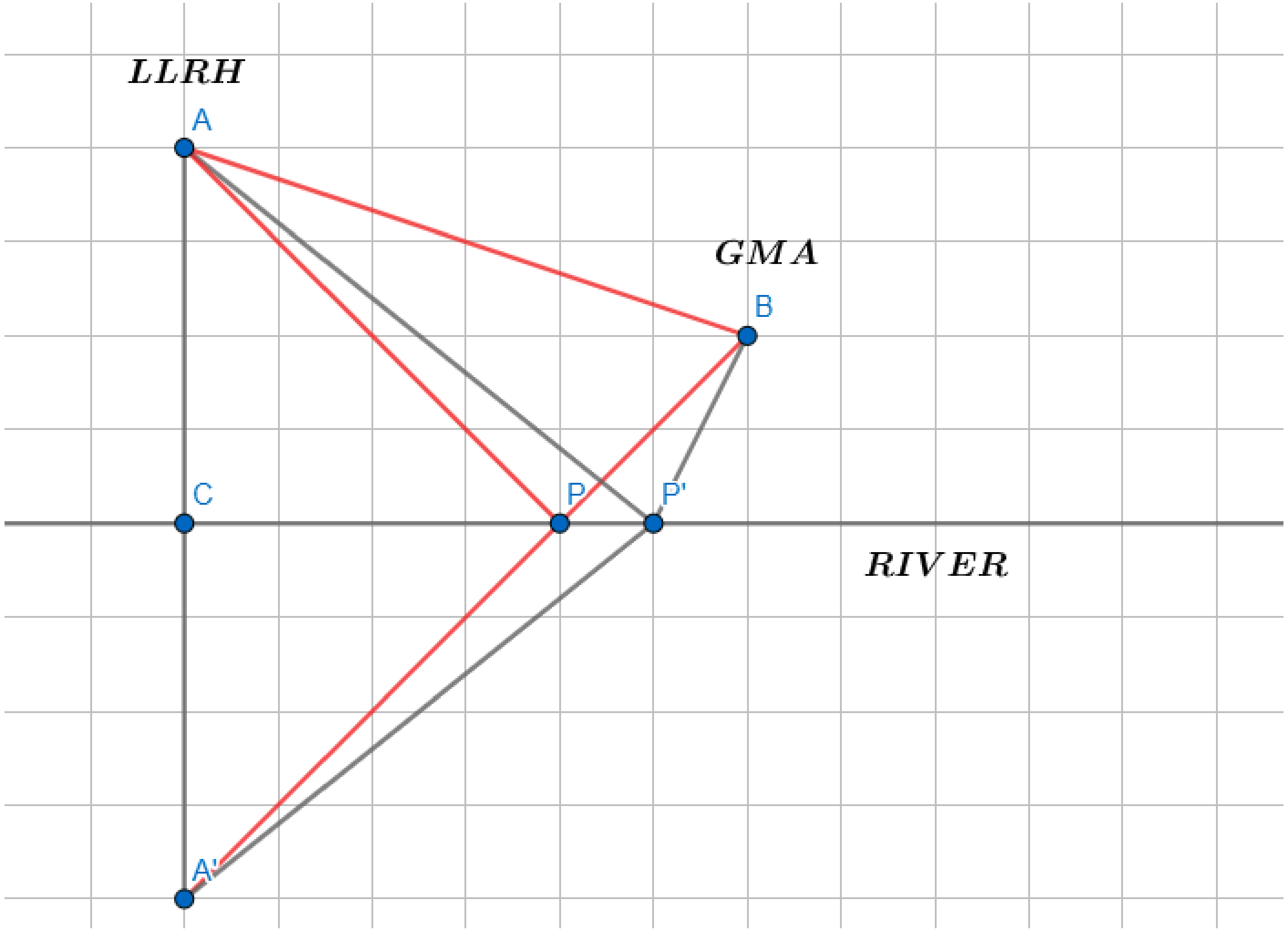
More Triangle Inequality Problems

- Little Red Riding Hood

Little Red Riding Hood leaves her house to bring some water to her old grandma. First she needs to go to the river to get some water. How should she do this so that her overall path, from home to the river and then to her grandma's house is as short as possible? Both LLRH and her grandma live on the same shore of the river. More precisely: you have two points A and B in the same half-plane of line l. Find a point P on l so that $AP + PB$ is the smallest possible.

Little Red Riding Hood

- P’ is an arbitrary point.
- Point C is the intersection of the segment connecting A and A’ with the river
- First you reflect A over the line to get A’.
- Then connect with A’ with B. The intersection of that segment and the river is point P.
- Point P is the location where LLRH should get the water. (Ultimately we want to show that any other point would be worse)



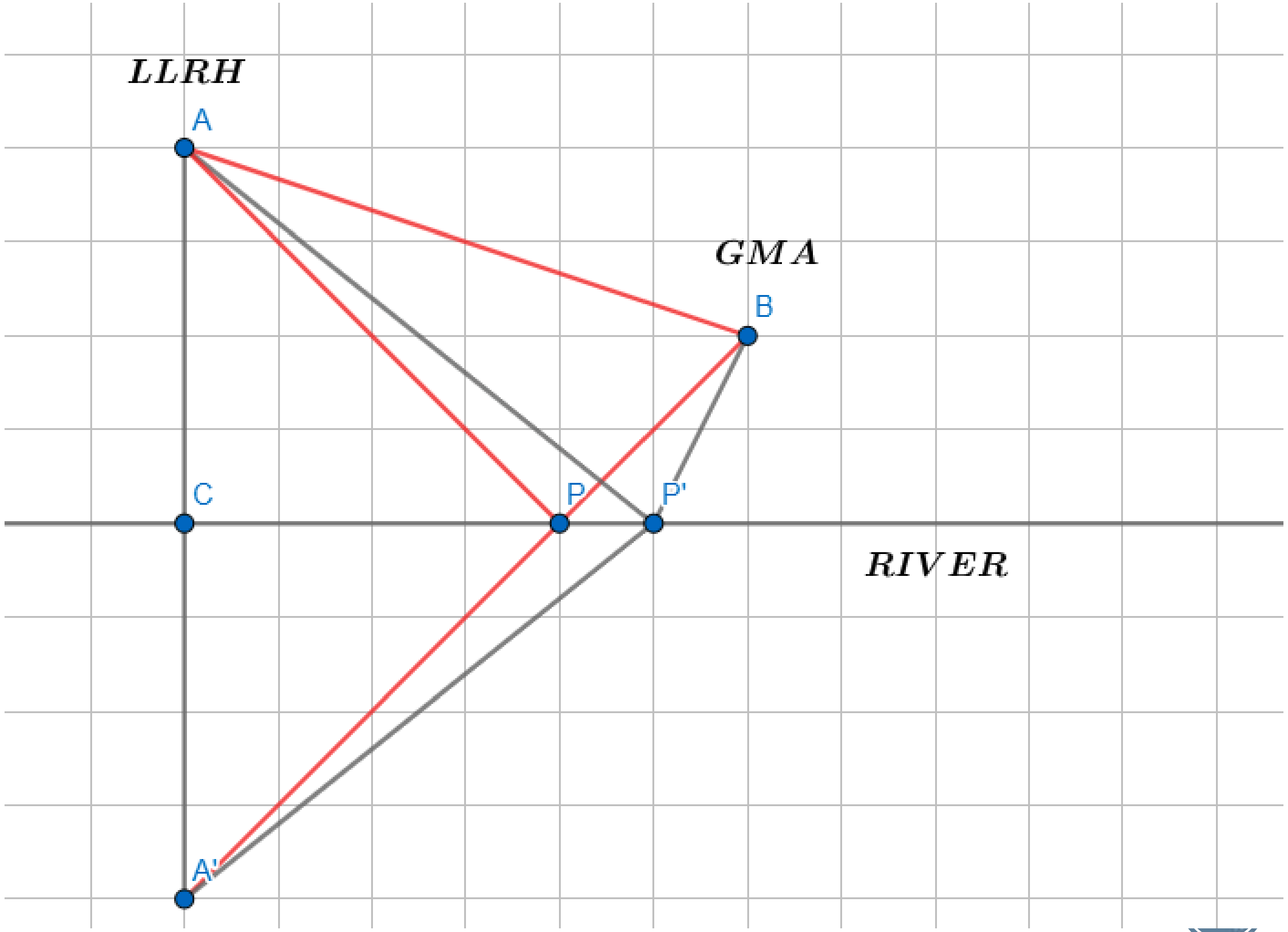
Little Red Riding Hood

$WTS: AP + PB \leq AP' + P'B$

$A'P + PB \leq A'P' + P'B$

$A'B \leq A'P' + P'B$

This holds true because of the triangle inequality



Investigating Further…

- A Stronger Triangle Inequality
  - Herbert R. Bailey & Robert Bannister

$$a + b > c + h$$

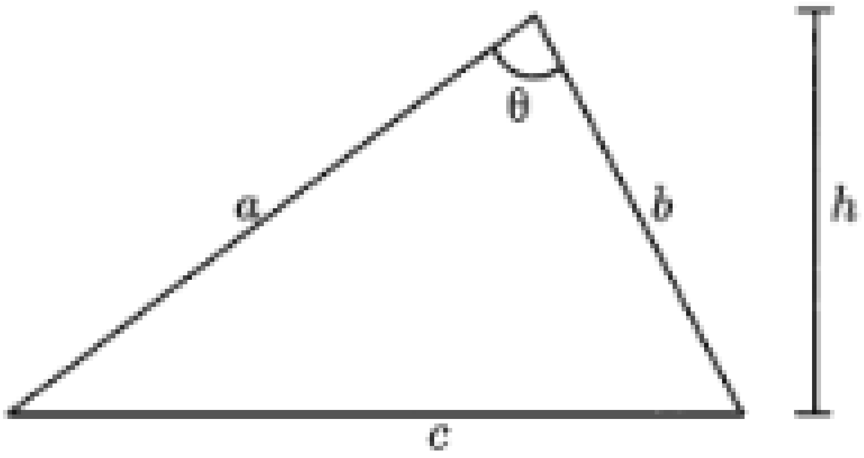


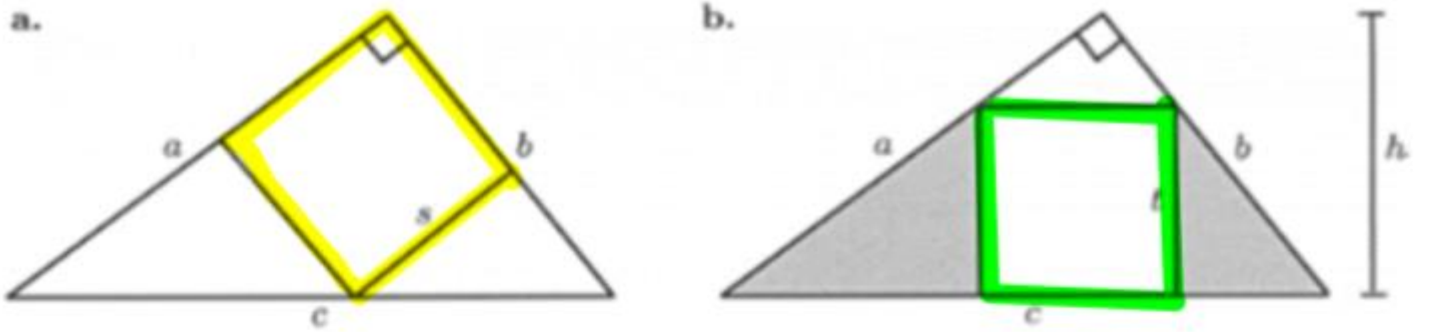
Figure 1

Want to show that the inequality works for most triangles where θ<$\frac{\pi}{2}$, a, b, c are sides of a triangle, and h is the altitude.

When $\theta = \frac{\pi}{2}$

The stronger triangle inequality is related to the following question… Which of the two squares has a larger area?

When investigating this question, $a + b > c + h$ is false when $\theta = \frac{\pi}{2}$.



$$\frac{s}{b-s} = \frac{a}{b} \Rightarrow s = \frac{ab}{a+b} \quad \text{and} \quad \frac{t}{c-t} = \frac{h}{c} \Rightarrow t = \frac{hc}{c+h}.$$

When $\theta = \frac{\pi}{2}$ continued…

$$\frac{s}{b-s} = \frac{a}{b} \Rightarrow s = \frac{ab}{a+b} \qquad \text{and} \qquad \frac{t}{c-t} = \frac{h}{c} \Rightarrow t = \frac{hc}{c+h}.$$

The numerators are equal to each other since
Area of $\Delta = \frac{1}{2}ab$ and Area of $\Delta = \frac{1}{2}hc$

Therefore, let’s compare the denominators…

When $\theta = \frac{\pi}{2}$ continued…

Finding the difference of the squares of the denominators and using substitution, we conclude that

$a + b < c + h$ $(contradicting\ the\ stronger\ triangle\ inequality)$ $and\ therefore\ s > t.$

Since $s > t, the\ first\ square\ is\ larger\ than\ the\ second\ one$.

Let's see if a+b>c+h is true for other values of θ.

- Let's compare $a+b$ and $c+h$
- Note: $Q \coloneqq (a+b)^2-(c+h)^2 > 0$ iff $a+b > c+h$
- Using law of cosines and using the two area formulas for the triangle, we can get the equations in terms of $a, b, and\ \theta$.
  - $Q = ab\left(2 + 2cos\theta - 2sin\theta - \frac{absin^2\theta}{a^2+b^2-2abcos\theta}\right)$

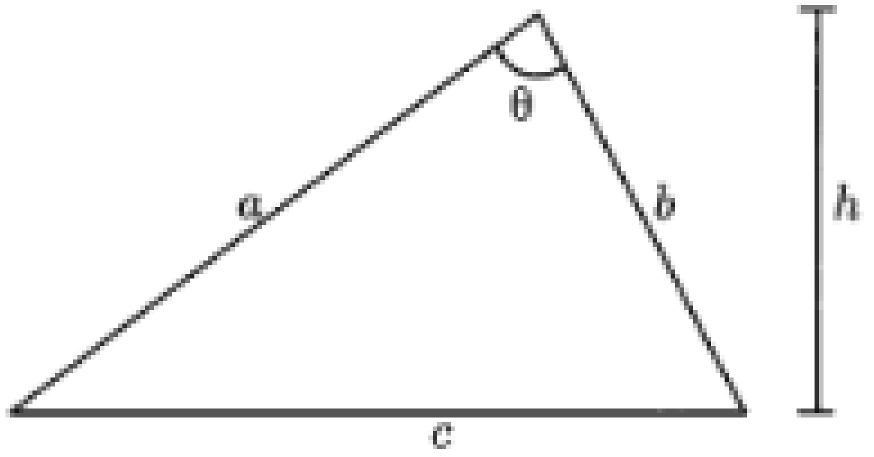


Figure 1

Let’s see if a+b>c+h is true for other values of θ.

- $Q = ab\left(2 + 2cos\theta - 2sin\theta - \frac{absin^2\theta}{a^2+b^2-2abcos\theta}\right)$
- Dividing the last term by b^2 and substituting $g(\theta) = 2(1 + cos\theta - sin\theta)$ and $R = \frac{a}{b}$.
- $Q = ab\left(g(\theta) - \frac{Rsin^2\theta}{R^2+1-2Rcos\theta}\right) = \frac{ab}{R^2+1-2Rcos\theta}(F(\theta, R))$
- Continuing to simplify we get the function
  - $F(\theta, R) = g(\theta)R^2 - [2g(\theta)cos\theta + sin^2\theta]R + g(\theta)$
  - This is a quadratic in R. $(AR^2 + BR + C)$
    - $A = g(\theta)$
    - $B = 2g(\theta)cos\theta + sin^2\theta$
    - $C = g(\theta)$
- Finding the discriminant $\mathcal{D}$=$(2g(\theta))(cos\theta + sin^2\theta - 2g(\theta))$
- Since $g(\theta) = 2(1 + cos\theta - sin\theta)$ we can substitute in and factor to get:
    - $\mathcal{D}$=$(1 - cos\theta)(4sin\theta - 3cos\theta - 3)(2 + 2cos\theta - sin\theta)^2$

Let's see if a+b>c+h is true for other values of θ.

$$F(\theta, R) > 0 \text{ iff}$$

$$\mathrm{Q} > 0 \text{ iff}$$

$$\mathrm{a} + \mathrm{b} > \mathrm{c} + \mathrm{h}$$ (the stronger triangle inequality)

- Points where the discriminant equals 0 (1 intersection)-show change in direction/branching out of curve.
- Used to help graph the function $z = F(\theta, R)$ and where it intersects in the (θ, R)/z=0 plane. The surface is either above or below the z=0 plane.
- The stronger triangle inequality holds iff $F(\theta, R)$ *is positive* $\rightarrow$ *above the* (θ, R) *plane*.
- Looking at Figure 6, we can see the function is above the plane from $0\ (P_1)$ *to* $\arctan\frac{24}{7} \approx 74°\ (P_2)$ which means the stronger triangle inequality holds (left of the curve)
- If $\theta = \frac{\pi}{2}$ if works only if $R = 0$; however R can never equal 0 since $R = \frac{a}{b}$; thus the stronger triangle inequality does not hold when $\theta = \frac{\pi}{2}$ (below the plane)

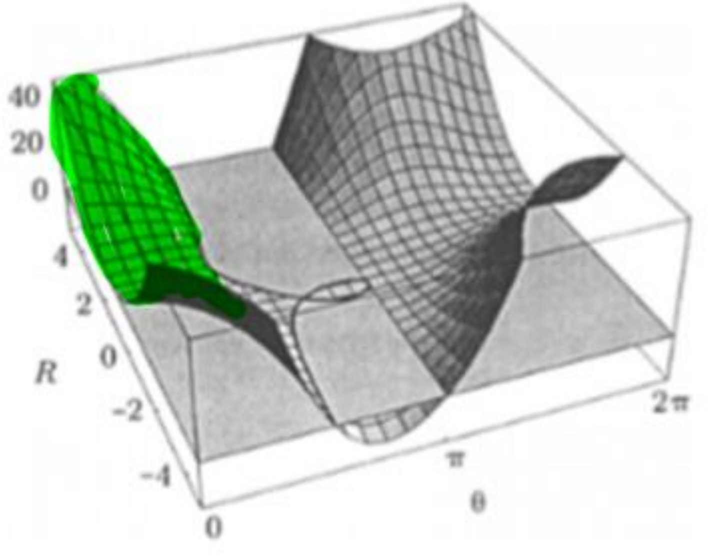


Figure 6. Graph of the function $z = F(\theta, R)$, with the plane $z = 0$.

$$P_1 = (0,1)$$

$$P_2 = (\arctan\frac{24}{7}, 1)$$

$$P_3 = (\frac{\pi}{2}, 0)$$

$$P_4 = (\pi - \arctan\frac{4}{3}, -1)$$

$$P_5 = (\pi, -1)$$

Curve= intersection of the surface with the z=0 plane.

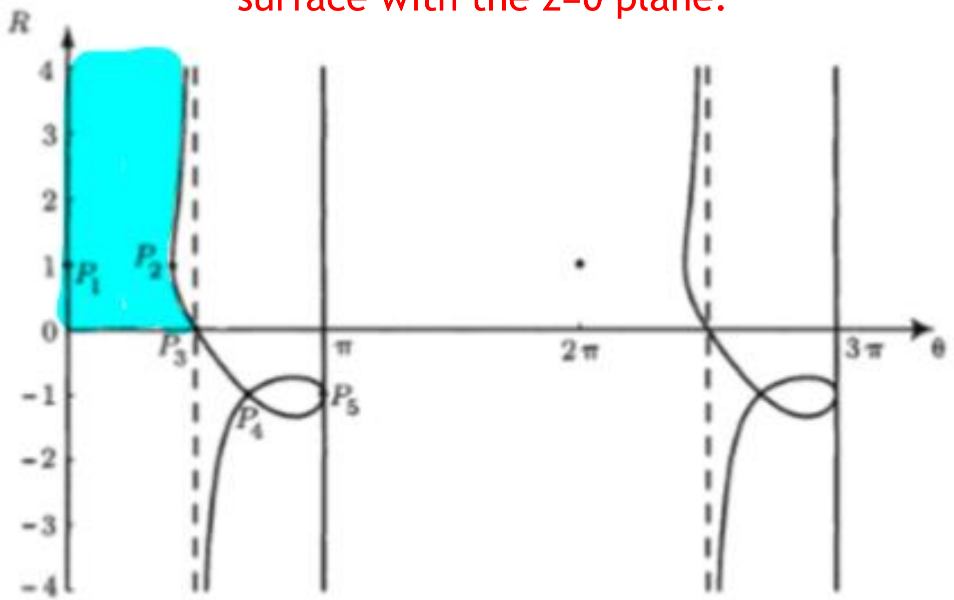


Figure 7. Graph of the level curve $F(\theta, R) = 0, 0 \leq \theta \leq 3\pi$.

References

- Bailey, H., & Bannister, R. (1997). A Stronger Triangle Inequality. *The College Mathematics Journal*, *28*(3), 182-186. doi:10.2307/2687521
- **Criminal Geometry, or A Matter of Princip**le (Sherlock Holmes displays math prowess), D. V. Fomin, Sep/Oct91, p46 (Smiles)